

A Step Block Algorithm for Solving First Order Initial Value Problems in Ordinary Differential Equations

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Abstract— The implementation of the newly formulated polynomials, ADEM-B orthogonal polynomials, $q_n(x)$ valid in the interval $[-1, 1]$ with respect to weight function $w(x) = x^2 - 1$ is our major focus in this work. The polynomials, which serve as basis function are employed to develop finite difference methods. Varying off-step points are considered for only One-Step method for the solution of the initial value problems of Ordinary Differential Equations (ODEs). By selection of points for both interpolation and collocation, three important classes of block finite difference methods are produced. The methods are analyzed for their basic properties and findings show that they are accurate and convergent.

Keywords— Orthogonal polynomials, Algorithm, Collocation, Interpolation, Zero-Stable

1 INTRODUCTION

First order differential equations arise in many important area of physical problems. The attempt to solve this type of equations led gradually to mathematical models involving an equation in which a function and its derivatives play important roles. The difficulties encounter in solving such problems has led to development of numerical methods. To develop such numerical methods, polynomial plays an important role. Notable among the well-known polynomials are the orthogonal polynomials. The first orthogonal polynomials were the Legendre polynomials. This was followed by the Chebyshev polynomials, the general Jacobi polynomials, the Hermite and the Laguerre polynomials. All these classical orthogonal polynomials play an important role in many applied problems. Asymptotic formulae for orthogonal polynomials were first discovered by Szego (1975). Lanczos (1938) introduced Chebyshev polynomials as trial function. Several researchers have employed these polynomials as trial functions to formulate algorithms (see Shampine and Watts (1969), Tanner (1979), Dahlquist (1979), Jator (2007) and Awoyemi (1991)).

Our focus in this paper shall be to apply the polynomials constructed in our earlier work (Adeyefa et al., 2016), which shall be succinctly presented in the next section to construct a set of one-step block methods for the solution of first order initial value problems.

2 CONSTRUCTION OF ORTHOGONAL BASIS FUNCTION

We define the orthogonal polynomial of the first kind of degree n over the interval $[-1, 1]$ with respect to weight

function $w(x) = x^2 - 1$ as

$$q_r(x) = \sum_{r=0}^n C_r^{(n)} x^r \quad (1)$$

The following requirements are considered:

$$\langle q_m(x), q_n(x) \rangle = 0, \quad m=0,1,2,\dots, n-1 \quad (2a)$$

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For the purpose of constructing the basis function, we adopt the approach discussed extensively in Adeyefa and Adeniyi (2015) and use additional property (the normalization)

$$q_n(1) = 1 \quad (2b)$$

Using (2), equation (1) yields the class of orthogonal polynomials presented in Adeyefa et al (2016) with therecurrence relation given as

$$P_{n+1}(x) = \frac{1}{n+3} [(2n+3)xP_n(x) - nP_{n-1}(x)], \quad \left. \begin{matrix} n \geq 1, P_0(x) = 1, P_1(x) = x \end{matrix} \right\} \quad (3)$$

3 FORMULATION OF THE ONE-STEP HYBRID METHODS

In this section, our aim is to derive one-step continuous schemes with different off-step points. To make this happen, we shall seek an approximant

$$y(x) = \sum_{r=0}^{s+k-1} a_r q_r(x) \quad (4)$$

to obtain the solution of first order initial value problems in ordinary differential equations. Transforming $q_n(x)$ to

$$x = \frac{2X - 2x_n - ph}{ph}$$

interval $[0, 1]$, we have $\frac{ph}{ph}$, where p varies as the method to be developed. In this case, $p = 1$, s and k in (4) are points of interpolation and collocation respectively. In what immediately follows, collocation at the both grid and off-grid points are considered with step size, $h = x_{i+1} - x_i$.

3.1 A One-Step Method with One Off-Step Point (OMOOP)

The procedure involves interpolating (4) at points $s = 0$ and collocating the first derivative of (4) at points $k = 0, \frac{1}{2}, 1$. The a_r , $r = 0(1)$ from the resulting system of equations are obtained as

$$a_o(t) = \phi_0 y_n + h(\phi_0 f_n + \phi_{\frac{1}{2}} f_{n+\frac{1}{2}} + \phi_1 f_{n+1})$$

$$a_r(t) = h(\phi_0 f_n + \phi_{\frac{1}{2}} f_{n+\frac{1}{2}} + \phi_1 f_{n+1}), \quad r = 1, 2, 3 \quad (5)$$

Substituting (5) into (4) yields the continuous implicit method

$$y(x) = \alpha_0 y_n + h \left(\sum_{j=0}^2 \beta_j f_{n+i} + \beta_{\frac{1}{2}} f_{n+\frac{1}{2}}, i=0,1 \right) \quad (6)$$

Evaluating equation (6) at $x = x_{n+m}$, $m=\frac{1}{2}, 1$ yields the discrete equations

$$\left. \begin{aligned} y_{n+\frac{1}{2}} &= y_n + \frac{h}{24} (5f_n + 8f_{n+\frac{1}{2}} - f_{n+1}) \\ y_{n+1} &= y_n + \frac{h}{6} (f_n + 4f_{n+\frac{1}{2}} + f_{n+1}) \end{aligned} \right\} \quad (7)$$

Equation (7) is our desired block method popularly called Gragg-Stetter method when evaluated at x_{n+1} only.

3.2 A One-Step Method with Two Off-Step Points (OMTOP)

Here, we interpolate(4) at $x=x_n$ and collocate the derivative of (4) at $x = x_n, x = x_{n+\frac{1}{3}}, x = x_{n+\frac{2}{3}}, x = x_{n+1}$

to obtain the following system of equations written in matrix form

$$\begin{bmatrix} 1 & -\frac{1}{2} & \frac{1}{3} & -\frac{1}{4} & \frac{1}{5} \\ 0 & \frac{3}{2} & -\frac{8}{3} & \frac{15}{4} & -\frac{24}{5} \\ 0 & \frac{3}{2} & -\frac{4}{3} & \frac{5}{4} & \frac{4}{5} \\ 0 & \frac{3}{2} & -\frac{9}{9} & \frac{6}{6} & \frac{5}{5} \\ 0 & \frac{3}{2} & \frac{16}{9} & \frac{5}{12} & -\frac{16}{15} \\ 0 & \frac{3}{2} & \frac{4}{4} & \frac{15}{12} & 12 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} = \begin{bmatrix} y_n \\ hf_n \\ hf_{n+\frac{1}{3}} \\ hf_{n+\frac{2}{3}} \\ hf_{n+1} \end{bmatrix} \quad (8)$$

Solving (8) by Guassian elimination method, we obtain the values of the a_k 's which are substituted into (4) to give the continuous implicit hybrid scheme

$$y(x) = \alpha_0(t) + h\beta_0(t)f_n + \beta_{\frac{1}{3}}(t)f_{n+\frac{1}{3}} + \beta_{\frac{2}{3}}(t)f_{n+\frac{2}{3}} + \beta_1(t)f_{n+1} \quad (9)$$

with parameters $\alpha(t)$ and $\beta(t)$ given as

$$\alpha_0(t) = 1, \beta_0(t) = 0,$$

$$\beta_{\frac{1}{3}}(t) = \frac{27}{128}t^4 - \frac{3}{32}t^3 - \frac{27}{64}t^2 + \frac{9}{32}t + \frac{51}{128}$$

$$\beta_{\frac{2}{3}}(t) = -\frac{27}{128}t^4 - \frac{3}{32}t^3 + \frac{27}{64}t^2 + \frac{9}{32}t - \frac{3}{128},$$

$$\beta_1(t) = \frac{9}{128}t^4 - \frac{3}{32}t^3 - \frac{1}{64}t^2 - \frac{1}{32}t + \frac{1}{128}$$

$$\text{where } t = \frac{2(x-x_n)-h}{h}$$

Evaluating (9) at $x = x_n, x = x_{n+\frac{1}{3}}, x = x_{n+\frac{2}{3}}, x = x_{n+1}$

we obtain a system of equation in the form

$$y_m = hPf_m + Ey_n + hDf_n$$

where

$$y_m = \begin{bmatrix} y_{n+\frac{1}{3}} \\ y_{n+\frac{2}{3}} \\ y_{n+1} \end{bmatrix}, P = \begin{bmatrix} \frac{19}{72} & -\frac{5}{72} & \frac{1}{72} \\ \frac{4}{9} & \frac{1}{9} & 0 \\ \frac{3}{8} & \frac{3}{8} & \frac{1}{8} \end{bmatrix}, E = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}, D = \begin{bmatrix} \frac{1}{8} \\ \frac{1}{9} \\ \frac{1}{8} \end{bmatrix}$$

3.3 A One-Step Method with Three Off-Step Points (OMTHOP)

We interpolate (4) at $x = x_n$ and collocate its derivative at

$$x = x_n, x = x_{n+\frac{1}{4}}, x = x_{n+\frac{1}{2}}, x = x_{n+\frac{3}{4}} \text{ and } x = x_{n+1} \text{ to}$$

obtain a system of equations

$$\begin{bmatrix} 1 & -\frac{1}{2} & \frac{1}{3} & -\frac{1}{4} & \frac{1}{5} & -\frac{1}{6} \\ 0 & \frac{3}{2} & -\frac{8}{3} & \frac{15}{4} & -\frac{24}{5} & \frac{35}{6} \\ 0 & \frac{3}{2} & -1 & -\frac{15}{64} & \frac{39}{40} & -\frac{175}{256} \\ 0 & \frac{3}{2} & \frac{2}{3} & -\frac{15}{6} & \frac{3}{35} & \frac{48}{35} \\ 0 & \frac{3}{2} & \frac{7}{7} & \frac{105}{105} & -\frac{3}{3} & \frac{1085}{1085} \\ 0 & \frac{3}{2} & \frac{4}{4} & \frac{15}{40} & -\frac{40}{768} & \frac{35}{2} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix} = \begin{bmatrix} y_n \\ hf_n \\ hf_{n+\frac{1}{4}} \\ hf_{n+\frac{1}{2}} \\ hf_{n+\frac{3}{4}} \\ hf_{n+1} \end{bmatrix} \quad (10)$$

Solving(10) by Guassian elimination method, we obtain the values of the a_k 's which are substituted into (4) to give the continuous implicit hybrid scheme as

$$y(x) = \alpha_0(t) + h\beta_0(t)f_n + \beta_{\frac{1}{4}}(t)f_{n+\frac{1}{4}} + \beta_{\frac{1}{2}}(t)f_{n+\frac{1}{2}} + \beta_{\frac{3}{4}}(t)f_{n+\frac{3}{4}} + \beta_1(t)f_{n+1} \quad (11)$$

Equation (11) gives $\alpha(t)$ and $\beta(t)$ as continuous functions of t as;

$$\alpha_0(t) = 1$$

$$\beta_0(t) = \frac{1}{15}t^5 - \frac{1}{12}t^4 - \frac{1}{36}t^3 + \frac{1}{24}t^2 + \frac{29}{360}$$

$$\beta_{\frac{1}{4}}(t) = -\frac{4}{15}t^5 + \frac{1}{6}t^4 + \frac{4}{9}t^3 - \frac{1}{3}t^2 + \frac{31}{90}$$

$$\beta_{\frac{1}{2}}(t) = \frac{2}{5}t^5 - \frac{5}{6}t^3 + \frac{1}{2}t^2 + \frac{1}{15}$$

$$\beta_{\frac{3}{4}}(t) = -\frac{4}{15}t^5 - \frac{1}{6}t^4 + \frac{4}{9}t^3 + \frac{1}{3}t^2 + \frac{1}{90}$$

$$\beta_1(t) = \frac{1}{15}t^5 + \frac{1}{12}t^4 - \frac{1}{36}t^3 - \frac{1}{24}t^2 - \frac{1}{360} \quad (12)$$

$$\text{where } t = \frac{2(x - x_n) - h}{h}$$

Evaluating (11) at $x = x_{n+\frac{1}{4}}, x = x_{n+\frac{1}{2}}, x = x_{n+\frac{3}{4}}, x = x_{n+1}$

we obtain a system of equation in matrix form as

$$y_m = hPf_m + Ey_n + hDf_n$$

$$y_m = \begin{bmatrix} y_{n+\frac{1}{4}} \\ y_{n+\frac{1}{2}} \\ y_{n+\frac{3}{4}} \\ y_{n+1} \end{bmatrix}, P = \begin{bmatrix} \frac{323}{1440} & -\frac{11}{120} & \frac{53}{1440} & -\frac{19}{2880} \\ \frac{31}{90} & \frac{1}{15} & \frac{1}{90} & -\frac{1}{360} \\ \frac{51}{160} & \frac{9}{40} & \frac{21}{160} & -\frac{3}{320} \\ \frac{16}{45} & \frac{2}{15} & \frac{16}{45} & \frac{7}{90} \end{bmatrix},$$

$$E = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, D = \begin{bmatrix} \frac{251}{2880} \\ \frac{29}{360} \\ \frac{27}{320} \\ \frac{7}{90} \end{bmatrix}$$

4 ANALYSIS OF THE METHOD

4.1 Order and Error Constant

Following Henrici (1962), the approach adopted in Fatunla (1991, 1994) and Lambert (1973), we define the local truncation error associated with equation block formulae by the difference operator

$$L[y(x); h] = \sum_{j=0}^k \left[\alpha_j y(x_n + jh) - h^2 \beta_j f(x_n + jh) \right] \quad (13)$$

where $y(x)$ is an arbitrary function, continuously differentiable on $[a, b]$.

Expanding (13) in Taylor series about the point x , we obtain the expression

$$L[y(x); h] = C_0 y(x) + C_1 h y'(x) + C_2 h^2 y''(x) + \dots + C_{p+3} h^{p+3} y^{(p+3)}(x)$$

where the $C_0, C_1, C_2, C_p, \dots$ are obtained as

$$C_0 = \sum_{j=0}^k \alpha_j, \quad C_1 = \sum_{j=1}^k j \alpha_j, \quad C_2 = \frac{1}{2!} \sum_{j=1}^k j^2 \alpha_j,$$

$$C_q = \frac{1}{q!} \left[\sum_{j=1}^k j^q \alpha_j - q(q-1)(q-2) \sum_{j=1}^k \beta_j j^{q-3} \right]$$

In the spirit of Lambert (1973), equations linear multistep methods are of order p if $C_0=C_1=C_2=\dots=C_p=0$ and $C_{p+1} \neq 0$. The C_{p+1} is called the error constant and $C_{p+1} h^{p+1} y^{(p+1)}(x_n)$ is the principal local truncation error at the point x_n .

According to the definition above, the Table of order and error constants of One-Step Methods is given as Table 1.

Table 1: Order and error constants of One-Step Methods

Method	Order, p	Error Constant
OMOOP	3	$\frac{1}{384}, 0$
OMTOP	4	$-\frac{19}{174960}, -\frac{1}{21870}, -\frac{1}{6480}$
OMTHOP	5	$\frac{3}{655360}, \frac{1}{368640}, \frac{3}{655360}, 0$

4.2 Zero Stability of the Method

According to Lambert (1973), a linear multistep method is said to be zero-stable if no root of the first characteristic polynomial $\rho(R)$ has modulus greater than one and if every root of modulus one has multiplicity not greater than the order of the differential equation.

To analyze the zero-stability of the methods, we have the

vector notation form of column vectors $e = (e_1 \dots e_r)^T$,

$d = (d_1 \dots d_r)^T$, $y_m = (y_{n+1} \dots y_{n+r})^T$,

$F(y_m) = (f_{n+1} \dots f_{n+r})^T$ and matrices $A = (a_{ij})$,

$B = (b_{ij})$.

Thus, the block methods form the block formula

$$A^0 y_m = hBF(y_m) + A^1 y_n + hDf_n \quad (14)$$

where h is a fixed mesh size within a block.

Hence, based on the definition above, the four schemes developed have been investigated to be zero stable.

4.3 Consistency of the Method

According to Lambert (1973), a linear multistep method is said to be consistent if it has order at least one. Owing to this definition, the One-Step Methods constructed are of order three, four and five. Thus, the schemes are consistent.

4.4 Convergence of the Method

According to the theorem of Dahlquist, the necessary and sufficient condition for a LMM to be convergent is to be consistent and zero stable. Since the methods satisfy these two conditions hence, they are convergent.

4.5 Numerical Experiment

The following examples are considered to implement the derived scheme.

Problem 1: $y' = 1 + y^2, y(0) = 0, h = 0.01$,

Exact solution: $y(x) = \tan x$

Table 1: Comparison of Error for Problem 1

X	OMOOP	OMTOP	OMTHOP
0.01	5.558e-13	3.4572e-12	6.7362e-12
0.02	1.1116e-12	1.43294e-11	3.01544e-11
0.03	1.6673e-12	3.26397e-11	7.03047e-11
0.04	2.2234e-12	5.84296e-11	1.272766e-10
0.05	2.7792e-12	9.17568e-11	2.011988e-10
0.06	3.3357e-12	1.326973e-10	2.922383e-10
0.07	3.8923e-12	1.813437e-10	4.006017e-10
0.08	4.4494e-12	2.378066e-10	5.265376e-10
0.09	5.0065e-12	3.022165e-10	6.703345e-10
0.1	5.564e-12	3.74722e-10	8.32325e-10

Problem 2: $y' = -y$, $y(0) = 1$, $h = 0.1$,

Exact Solution: $y(x) = e^{-x}$

Table 2: Comparison of Error for Problem 2

X	OMOOP	OMTOP	OMTHOP
0.1	1.2574666e-8	4.4204169e-7	1.00267323e-6
0.2	2.2756058e-8	7.999513e-7	1.81451150e-6
0.3	3.0885799e-8	1.0857388e-6	2.46275549e-6
0.4	3.726217e-8	1.3098892e-6	2.97118945e-6
0.5	4.2145258e-8	1.48154559e-6	3.36055237e-6
0.6	4.5761528e-8	1.608669e-6	3.64890222e-6
0.7	4.8307866e-8	1.698189e-6	3.85193834e-6
0.8	4.995516e-8	1.756088e-6	3.98328687e-6
0.9	5.0851416e-8	4.420417e-7	4.05475314e-6
1.0	5.1124784e-8	7.999513e-7	4.07654481e-6

Problem 3: $y'(x) = -9y$, $y(0) = e$, $h = 0.1$,

Exact Solution: $y(x) = e^{1-9x}$

Table 3: Comparison of Error for Problem 3

X	OMOOP	OMTOP	OMTHOP
0.1	9.50353e-4	1.10107e-4	1.17009e-6
0.2	7.73101e-4	8.95275e-5	1.38315e-6
0.3	4.71682e-4	5.45960e-5	8.43519e-7
0.4	2.55805e-4	2.95946e-5	4.57266e-7
0.5	1.30059e-4	1.50396e-5	2.32388e-7
0.6	7.34922e-5	7.33721e-6	1.13379e-7
0.7	3.01241e-5	3.48009e-6	5.37790e-8
0.8	1.40032e-5	1.61695e-6	2.49885e-8
0.9	6.40769e-6	7.39541e-7	1.14295e-8
1.0	2.89588e-6	3.34067e-7	5.16322e-9

5 CONCLUSION

Construction of continuous solutions to first order initial value problems over a discretized interval by one-step collocation has been considered. We have used the new class of polynomial earlier constructed to formulate the popular Gragg-Stetter method (presented in block method) and two other finite difference methods. Three test problems have been considered to show the efficiency and accuracy of the methods. In our future paper, we hope to extend this approach to solve partial differential equations as well as a consideration of global error estimates based on the solutions.

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